

Spontaneous Dimensional Reduction in Short-Distance Quantum Gravity?

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Abstract. Several lines of evidence suggest that quantum gravity at very short distances may behave effectively as a two-dimensional theory. I summarize these hints, and offer an additional argument based on the strong-coupling limit of the Wheeler-DeWitt equation. The resulting scenario suggests a novel approach to quantum gravity at the Planck scale.

Keywords: quantum gravity, Planck scale, dimensional reduction

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At large scales, spacetime behaves as a smooth four-dimensional manifold. At the Planck scale, on the other hand, the appropriate description is not so clear: we have neither observational evidence nor an established theoretical framework, and it is not even obvious that “space” and “time” are proper categories.

But while a complete quantum theory of gravity remains distant, we have a number of fragments that may offer hints. When these fragments fit together—when a fundamental feature of spacetime appears robustly across different approaches to quantum gravity—we should consider the possibility that our models are telling us something real about Nature. The thermodynamic behavior of black holes, for example, occurs so consistently, across so many different approaches, that it is reasonable to expect quantum gravity to provide a statistical mechanical explanation.

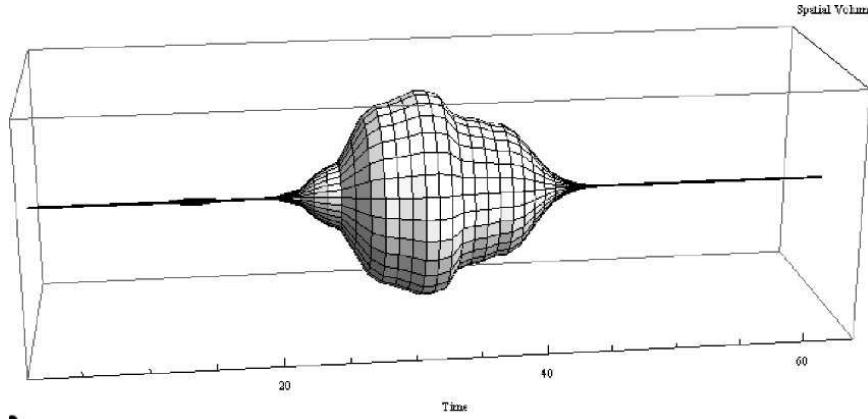
Over the past few years, evidence has accumulated that spacetime near the Planck scale is effectively two-dimensional. No single indication of this behavior is in itself very convincing, but taken together, they may point toward a promising direction for further investigation. Here, I will summarize these hints, and provide a new piece of evidence in the form of a strong-coupling approximation to the Wheeler-DeWitt equation.

IS SMALL-SCALE QUANTUM GRAVITY TWO-DIMENSIONAL?

Evidence for “spontaneous dimensional reduction” at short distances comes from a variety of different approaches to quantum gravity. Among these are the following:

Causal Dynamical Triangulations

The “causal dynamical triangulation” program [1, 2, 3] is a discrete approximation to the gravitational path integral, in which the spacetimes contributing to the sum over histories are approximated by locally flat simplicial manifolds. The idea of a simplicial



Source: R. Kommu, U.C. Davis

FIGURE 1. A contribution to the path integral in causal dynamical triangulations

approximation dates back to Regge's work in the 1960s [4], and the suggestion of using Monte Carlo calculations was made as early as 1981 [5]. Until fairly recently, though, such efforts failed, yielding only a "crumpled" phase with a very high Hausdorff dimension and a two-dimensional "branched polymer" phase [6]. The crucial new ingredient introduced by Ambjørn et al. is a definite causal structure, in the form of a fixed time-slicing. The resulting path integral appears to lead to four-dimensional spacetimes, with contributions of the form shown in figure 1 [7]; moreover, the computed cosmological scale factor has the correct semiclassical behavior [3, 8].

A key question for any such discrete approach is whether it genuinely reproduces the four-dimensional structure we observe at large distances. This is a subtle issue, requiring a definition of dimension for a discrete structure that may be very non-manifold-like at short distances. One natural choice is the spectral dimension [9], the dimension as seen by a diffusion process or a random walker.

Diffusion from an initial position x to a final position x' in time s may be described by a heat kernel $K(x, x', s)$, satisfying

$$\left(\frac{\partial}{\partial s} - \Delta_x \right) K(x, x'; s) = 0, \quad \text{with} \quad K(x, x', 0) = \delta(x - x'). \quad (1)$$

On a manifold of dimension d_S , the heat kernel generically behaves as

$$K(x, x'; s) \sim (4\pi s)^{-d_S/2} e^{-\sigma(x, x')/2s} (1 + \mathcal{O}(s)) \quad (2)$$

for small s , where $\sigma(x, x')$ is Syng's "world function" [10], one-half of the geodesic distance between x and x' . In particular, the return probability $K(x, x, s)$ is

$$K(x, x; s) \sim (4\pi s)^{-d_S/2}. \quad (3)$$

For any space on which a diffusion process can be defined, we can then use equation (3) to define an effective dimension d_S , the spectral dimension.

For the causal dynamical triangulation program, the spectral dimension is measured, to within numerical accuracy, to be $d_S = 4$ at large distances [3, 9]. This is a promising sign, indicating the recovery of four-dimensional behavior. At short distances, though, the spectral dimension falls to two. A similar behavior occurs in (2+1)-dimensional gravity [7]. This is our first indication of dimensional reduction at short distances.

Now, the spectral dimension is not the unique generalization of dimension, and one may worry about reading too much significance into this result. Note, though, that the propagator for a scalar field may be obtained as a Laplace transform of the heat kernel. The behavior of the spectral dimension then leads to a propagator

$$G(x, x') \sim \int_0^\infty ds K(x, x'; s) \sim \begin{cases} \sigma^{-2} & \text{at large distances} \\ \ln \sigma & \text{at small distances.} \end{cases} \quad (4)$$

The logarithmic short-distance behavior is characteristic of a two-dimensional field theory, and strongly suggests that if one probes short distances with quantum fields, one will measure an effective dimension of two.

Renormalization Group Analysis

General relativity is, of course, nonrenormalizable. Nevertheless, a renormalization group analysis may give us useful information about quantum gravity. In particular, Weinberg has suggested that the theory may be “asymptotically safe” [11].

Consider the full effective action for metric gravity, containing an infinite number of higher-derivative terms with an infinite number of coupling constants. Under the renormalization group flow, some of these constants may blow up, indicating that the effective action description has broken down and new physics is needed. It could be, however, that the coupling constants remain finite and flow to an ultraviolet fixed point. In that case, the theory would continue to make sense down to arbitrarily short distances. If, in addition, the critical surface—the space of such UV fixed points—were finite dimensional, the coupling constants would be determined by a finite number of parameters: not quite renormalizability, but almost as good.

We do not yet know whether quantum general relativity exhibits such behavior. But renormalization group flows of a variety of truncated actions offer evidence of a UV fixed point [12, 13, 14]. For the present investigation, the key feature of these results is that operators obtain large anomalous dimensions at the fixed point—precisely the dimensions that characterize a two-dimensional field theory [12]. Moreover, a computation of the spectral dimension near the putative fixed point again yields $d_S = 2$ [15].

There is, in fact, a fairly general argument that if quantum gravity is asymptotically safe, it must behave like a two-dimensional theory at the UV fixed point [14]. Consider the dimensionless coupling constant $g_N(\mu) = G_N \mu^{d-2}$, where G_N is Newton’s constant. Under renormalization group flow,

$$\mu \frac{\partial g_N}{\partial \mu} = [d - 2 + \eta_N(g_N, \dots)] g_N, \quad (5)$$

where the anomalous dimension η_N depends on both g_N and any other dimensionless

coupling constants in the theory. For a non-Gaussian fixed point g_N^* to occur,¹ the right-hand side of (5) must vanish, that is, $\eta_N(g_N^*, \dots) = 2 - d$.

But the propagator of a field with anomalous dimension η_N has a momentum dependence of the form $(p^2)^{-1+\eta_N/2}$. For $\eta_N = 2 - d$, this is p^{-d} , and the corresponding position space propagator depends logarithmically on distance. Such behavior is, again, the characteristic of a two-dimensional field. While the argument I have given applies to the graviton propagator, a generalization to arbitrary fields is straightforward [14].

Loop quantum gravity

Our next indication of short-distance dimensional reduction comes from the area spectrum of loop quantum gravity [16]. This spectrum is labeled by half-integers j : $A_j \sim \ell_p^2 \sqrt{j(j+1)}$, where ℓ_p is the Planck length. Defining $\ell_j = \sqrt{j} \ell_p$, we can rewrite the spectrum as

$$A_j \sim \sqrt{\ell_j^2(\ell_j^2 + \ell_p^2)} \sim \begin{cases} \ell_j^2 & \text{for large areas} \\ \ell_p \ell_j & \text{for small areas.} \end{cases} \quad (6)$$

Like the propagator (4), this spectrum undergoes a change in scaling at small distances. Modesto argues that this behavior determines the scaling of an effective metric, and uses this scaling to compute a spectral dimension. The result is again an effective dimension that decreases from four at large scales to two at small scales.

High temperature strings

Yet another piece of evidence comes from the high temperature behavior of string theory. In 1988, Atick and Witten showed that at temperatures far above the Hagedorn temperature, string theory has a very peculiar thermodynamic behavior [17]: the free energy in a volume V varies with temperature as

$$F/VT \sim T. \quad (7)$$

For a field theory in d dimensions, in contrast, $F/VT \sim T^{d-1}$. Thus, although string theory lives in 10 or 26 dimensions, at high temperatures it behaves in some ways as if spacetime were two-dimensional.

Anisotropic scaling models

As a final indication of spontaneous dimensional reduction, we can consider “Hořava-Lifshitz gravity” [18], a set of new models of gravity that exhibit anisotropic scaling, that is, invariance under rescalings $\mathbf{x} \rightarrow b\mathbf{x}$, $t \rightarrow b^3 t$. Such a scaling property clearly breaks Lorentz invariance (which may, however, be restored at low energies). In fact, this symmetry breaking is the key to renormalizability: the field equations may contain many spatial derivatives, giving high inverse powers of spatial momentum to tame loop integrals, while keeping only second time derivatives, thus avoiding ghosts.

¹ “Non-Gaussian” simply means “not free field,” i.e., $0 < g_N^* < \infty$.

Hořava has calculated the spectral dimension in such models [19], and finds that $d_S = 2$ at high energies. In one sense, this is a cautionary tale: the “two-dimensional” behavior arises from the fact that the propagators contain higher inverse powers of momentum, and the logarithmic dependence on distance comes from integrals of the form $\int d^4p/p^4$ rather than from any intrinsic two-dimensional structure. The lesson, I believe, is that “dimension” is not such an obvious quantity in quantum gravity, but may have different meanings depending on how one probes the physics. In particular, despite the four-dimensional origin of the spectral dimension in these models, the results imply that quantum fields will behave “two-dimensionally” at short distances.

THE STRONG-COUPING APPROXIMATION

Suppose the hints of the preceding section are really telling us something deep about short-distance quantum gravity. A number of obvious questions arise. Most strikingly, we may ask, “Which two dimensions?” How can a theory with a four-dimensional Lorentz symmetry pick out two “preferred” directions at small scales? To address this question, let us consider one more approach to physics at the Planck scale: the strong-coupling approximation of the Wheeler-DeWitt equation.

As early as 1976, Isham [20] noted that the Wheeler-DeWitt equation [21]

$$\left\{ 16\pi\ell_p^2 G_{ijkl} \frac{\delta}{\delta g_{ij}} \frac{\delta}{\delta g_{kl}} - \frac{1}{16\pi\ell_p^2} \sqrt{g} {}^{(3)}R \right\} \Psi[g] = 0 \quad (8)$$

has an interesting strong-coupling limit $\ell_p \rightarrow \infty$. In this limit, the Wheeler-DeWitt equation becomes ultralocal: spatial derivatives appear only in the scalar curvature term, and when this term drops out, points effectively decouple. As Pilati first observed [22], this limit probes spacetime near or below the Planck scale; Maeda and Sakamoto have expounded this argument in more detail [23].

The $\ell_p = \infty$ limit of the Wheeler-DeWitt equation was studied extensively in the 1980s [24, 25, 26, 27, 28] and a perturbative treatment of the scalar curvature term has been discussed by several authors [29, 30, 31]. The key features are already evident in the classical version. The strong-coupling approximation can also be viewed as a small c approximation; as ℓ_p becomes large, light cones contract to timelike lines, and neighboring points decouple [32]. The classical solution at each point is a Kasner space,

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2 \quad (9)$$

$$(-\frac{1}{3} < p_1 < 0 < p_2 < p_3, \quad p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2).$$

More precisely—see, for example, [33]—the general solution is an arbitrary GL(3) transformation of a Kasner metric: in effect, a Kasner space with arbitrary, not necessary orthogonal, axes.

For large but finite ℓ_p , the classical solution exhibits BKL behavior [34, 35]. At each point, the metric spends most of its time in a nearly Kasner form. But the scalar curvature can grow abruptly, making the curvature term in (8) important and leading to a Mixmaster-like “bounce” [36] to a new Kasner solution with different axes and

exponents. Neighboring points are no longer completely decoupled, but the Mixmaster bounces are chaotic [37]; the geometries at nearby points quickly become uncorrelated, with Kasner exponents occurring randomly with a known probability distribution [38].

We can now return to the problem of dimensional reduction. Consider a timelike geodesic in Kasner space, starting at $t = t_0$ with a randomly chosen initial velocity. It is not hard to show that in the direction of decreasing t , the proper distance along the geodesic in the direction of each of the Kasner axes asymptotes to

$$\begin{aligned} s_x &\sim t^{p_1} \\ s_y &\sim 0 \\ s_z &\sim 0. \end{aligned} \tag{10}$$

The geodesic effectively explores only one spatial dimension. In the direction of increasing t , a similar, though less dramatic, phenomenon occurs:

$$\begin{aligned} s_x &\sim t \\ s_y &\sim t^{\max(p_2, 1+p_1-p_2)} \\ s_z &\sim t^{p_3}. \end{aligned} \tag{11}$$

Since $\max(p_2, 1+p_1-p_2)$ and p_3 are both less than one, a random geodesic again preferentially sees one dimension of space.

One might expect this behavior to be reflected in the heat kernel and the spectral dimension. The exact form of heat kernel for Kasner space is not known, but Futamase [39] and Berkin [40] have looked at different approximations. Both find behavior of the form

$$K(x, x; s) \sim \frac{1}{4\pi s^2} \left[1 + \frac{a}{t^2} s + \dots \right]. \tag{12}$$

For a fixed time t , one can always find s small enough that the first term in (12) dominates: the heat kernel is a classical object, and the underlying classical spacetime is still four-dimensional. For a fixed return time s , on the other hand, one can always find a time t small enough that the second term dominates, leading to an effective spectral dimension of two.² The idea that the “effective infrared dimension” might differ from four goes back to work by Hu and O’Connor [41], but the relevance to short-distance quantum gravity was not fully appreciated at that time.

One can also investigate this issue by using the Seeley-DeWitt expansion of the heat kernel [42, 43, 44],

$$K(x, x; s) \sim \frac{1}{4\pi s^2} ([a_0] + [a_1]s + [a_2]s^2 + \dots). \tag{13}$$

The “Hamidew coefficient” $[a_1]$ is proportional to the scalar curvature, and vanishes for an exact vacuum solution of the field equations. In the presence of matter, however, the

² One might worry that at smaller t , even higher powers of s dominate. But these lead to terms in the propagator that go as positive powers of the geodesic distance, and are irrelevant for short distance singularities, light cone behavior, and the like.

scalar curvature will typically increase as an inverse power of t as $t \rightarrow 0$ [34]; this growth is slow enough to not disrupt the BKL behavior of the classical solutions near $t = 0$, but it will nevertheless give a diverging contribution to $[a_1]$.

The short-distance BKL behavior suggested by the Wheeler-DeWitt equation may thus offer an explanation for the dimensional reduction of quantum gravity at the Planck scale. The dynamics picks out an essentially random dominant spatial direction at each point, whose existence is reflected in the behavior of the heat kernel and the propagators. To probe this picture further, though, we must better understand the underlying physics.

ASYMPTOTIC SILENCE?

The BKL picture was originally developed in a very different context from the one I am considering here, as a study of the Universe near an initial spacelike singularity. In that setting, the key physical ingredient is “asymptotic silence” [35], the strong focusing of light by the singularity that collapses light cones and shrinks particle horizons. It is this behavior that decouples neighboring points and leads to the ultralocal form of the equations of motion.

The similar decoupling of neighboring points in the strongly coupled Wheeler-DeWitt equation suggests that a similar physical mechanism may be at work. To see whether this is the case will require much deeper investigation. As a first hint, though, consider the classical Raychaudhuri equation,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_\alpha{}^\beta \sigma_\beta{}^\alpha + \omega_{\alpha\beta} \omega^{\alpha\beta} - R_{\alpha\beta} k^\alpha k^\beta, \quad (14)$$

for the expansion of a bundle of light rays. We do not know the quantum version of this relation, but if we naively treat (14) as an operator equation in the Heisenberg picture and take the expectation value, we see that quantum fluctuations in the expansion and shear always focus geodesics:

$$\langle \theta^2 \rangle = \langle \theta \rangle^2 + (\Delta\theta)^2, \quad (15)$$

with a similar equation for σ .

How strong is this focusing? Roughly speaking, the expansion θ is canonically conjugate to the cross-sectional area of the congruence [45]: as the trace of an extrinsic curvature, it is conjugate to the corresponding volume element. Keeping track of factors of \hbar and G , one finds an uncertainty relation

$$\Delta\bar{\theta}\Delta A \sim \ell_p, \quad (16)$$

where $\bar{\theta}$ is the expansion averaged over a Planck distance along the congruence. If, near the Planck scale, the area uncertainty is of order ℓ_p^2 —as one might expect from theories such as loop quantum gravity in which the area spectrum is quantized—this would imply fluctuations of θ of order $1/\ell_p$, giving strong focusing.

This argument is, of course, only suggestive. In particular, I have ignored the need to renormalize products of operators such as θ^2 , and have neglected the effects of the

twist and curvature terms in (14). But while the result is not yet established, it is at least plausible that quantum fluctuations at the Planck scale, “spacetime foam,” could lead to strong focusing of geodesics, and thus to short-distance asymptotic silence.

A NEW PICTURE

If the proposals of the two preceding sections are correct—if spacetime foam strongly focuses geodesics at the Planck scale, leading to the BKL behavior predicted by the strongly coupled Wheeler-DeWitt equation—they suggest a novel picture of small-scale spacetime. At each point, the dynamics picks out a “preferred” spatial direction, leading to approximately (1+1)-dimensional local physics. The preferred directions are presumably determined by initial conditions, but because of the chaotic behavior of BKL bounces, they are quickly randomized. From point to point, these directions vary continuously, but oscillate rapidly [46]. Space at a fixed time is thus threaded by rapidly fluctuating lines, and spacetime by two-surfaces, and the leading behavior of the physics is described by an approximate “dimensional reduction” to these surfaces.

There is a danger here, of course: the process I have described breaks Lorentz invariance at the Planck scale, and even small violations at such scales can have observable effects at larger scales [47]. Note, though, that the symmetry violations in the present scenario vary rapidly and essentially stochastically in both space and time. Such “non-systematic” Lorentz violations are harder to study, but there is evidence that they lead to much weaker observational constraints [48].

The scenario I have presented is still very speculative, but I believe it deserves further investigation. One avenue might be to use results from the eikonal approximation [49, 50, 51]. In this approximation, developed to study very high energy scattering, a similar dimensional reduction takes place, with drastically disparate time scales in two pairs of dimensions. Although the context is very different, the technology developed for this approximation could prove useful for the study of Planck scale gravity.

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